Asymptotic Notations

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Review

- We can choose from a wide range of algorithm design techniques
  - Incremental Approach
    - Insertion Sort
  - Divide-and-conquer Approach
    - Merge Sort
    - One advantage of divide-and-conquer algorithms is that their running times are often easily determined

Asymptotic Notations

- We introduce some terminology that will enable is to make **meaningful but inexact** statements about the time and space complexities of a program

Definition [*Big "oh"*]: $f(n) = O(g(n))$ (read as "f of n is big oh of g of n") iff (if and only if) there exist positive constants c and n_0 such that $f(n) \leq cg(n)$ for all n, $n \geq n_0$. □

Definition: [Omega] $f(n) = \Omega(g(n))$ (read as "f of n is omega of g of n") iff there exist positive constants c and n_0 such that $f(n) \geq cg(n)$ for all n, $n \geq n_0$. □

Definition: [Theta] $f(n) = \Theta(g(n))$ (read as "f of n is theta of g of n") iff there exist positive constants c_1, c_2, and n_0 such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all n, $n \geq n_0$. □

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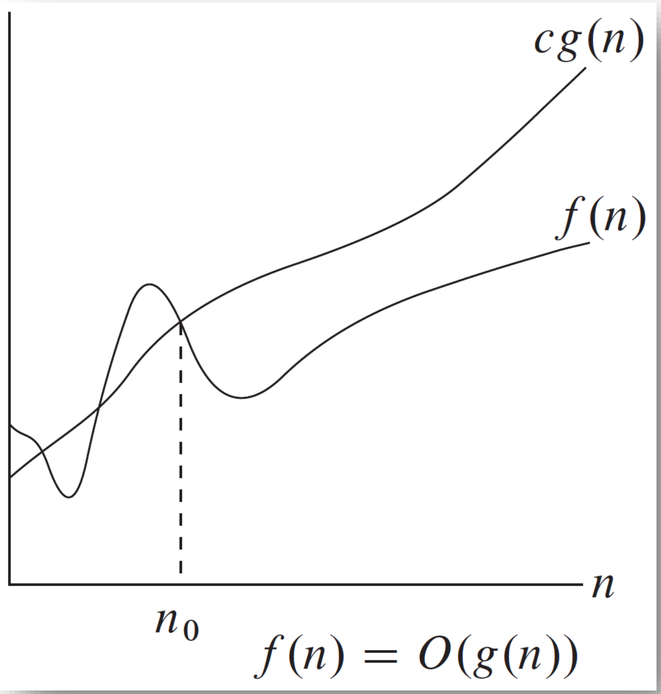
- $f(n) = O(g(n))$ means that $c \times g(n)$ is an **asymptotic upper bound** on the value of $f(n)$ for all n, where $n \geq n_0$



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Example 1.14: $3n + 2 = O(n)$ as $3n + 2 \le 4n$ for all $n \ge 2$. $3n + 3 = O(n)$ as $3n + 3 \le 4n$ for all $n \ge 3$. $100n + 6 = O(n)$ as $100n + 6 \le 101n$ for $n \ge 10$. $10n^2 + 4n + 2 = O(n^2)$ as $10n^2 + 4n + 2 \le 11n^2$ for $n \ge 5$. $1000n^2 + 100n - 6 = O(n^2)$ as $1000n^2 + 100n - 6 \le 1001n^2$ for $n \ge 100$. $6*2^n + n^2 = O(2^n)$ as $6*2^n + n^2 \le 7*2^n$ for $n \ge 4$. $3n + 3 = O(n^2)$ as $3n + 3 \le 3n^2$ for $n \ge 2$. $10n^2 + 4n + 2 = O(n^4)$ as $10n^2 + 4n + 2 \le 10n^4$ for $n \ge 2$. $3n + 2 \ne O(1)$ as $3n + 2$ is not less than or equal to c for any constant c and all n, $n \ge n_0$. $10n^2 + 4n + 2 \ne O(n)$. □

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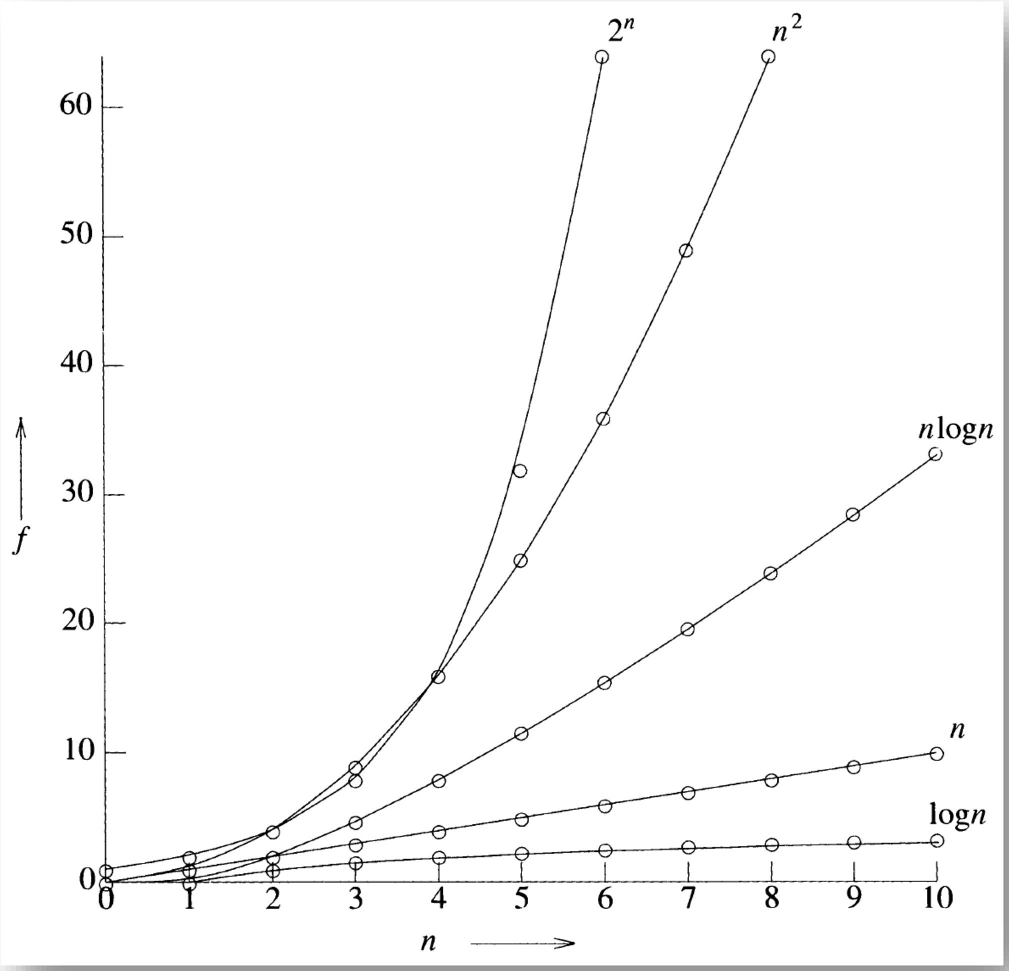
- For the statement $f(n) = O(g(n))$ to be **informative**, $g(n)$ should be as small a function of n as one can come up with
  - $3n + 3 = O(n)$ vs. $3n + 3 = O(n^2)$

- Fantastic names
  - $O(1)$ mean a computing time that is a constant
  - $O(n)$ is called linear
  - $O(n^2)$ is called quadratic
  - $O(n^3)$ is called cubic
  - $O(2^n)$ is called exponential

- Ordering
  - $O(1) < O(\log n) < O(n) < O(n\log n) < O(n^2) < O(n^3) < O(2^n)$

Big-Oh…

- $O(1) < O(\log n) < O(n) < O(n\log n) < O(n^2) < O(n^3) < O(n^c) < O(2^n) < O(3^n) < O(c^n) < O(n!) < O(n^n) < O\left(n^{c^n}\right)$



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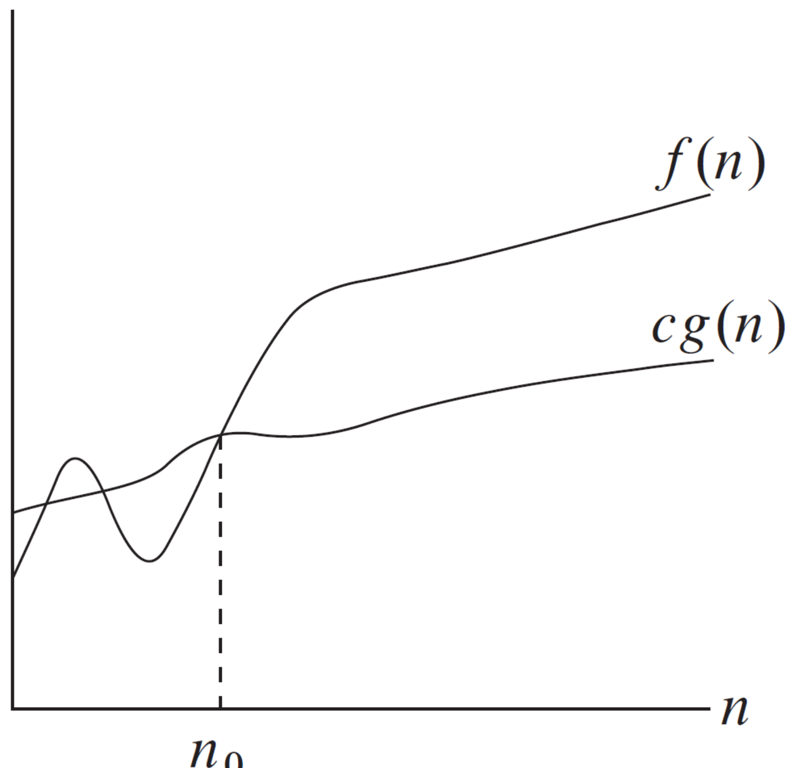

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- For the statement $f(n) = \Omega(g(n))$ to be informative, $g(n)$ should be as large a function of n as possible
  - $3n + 3 = \Omega(n)$ vs. $3n + 3 = \Omega(1)$
  - $6 \times 2^n + n^2 = \Omega(2^n)$ vs. $6 \times 2^n + n^2 = \Omega(1)$

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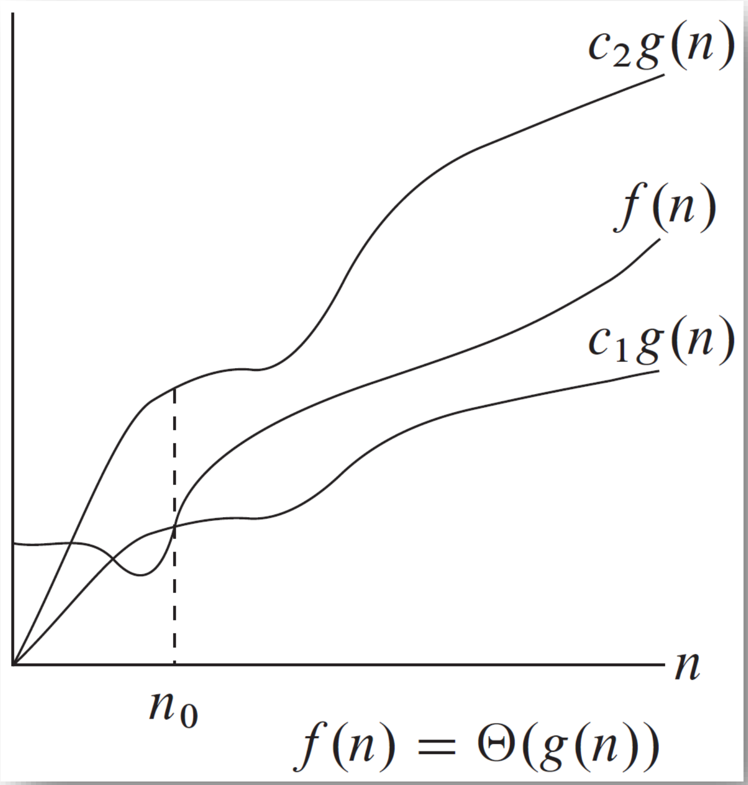
- The theta is more precise than both big-oh and omega
  - $g(n)$ is both an upper and lower bound on $f(n)$



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Example 1.16: $3n + 2 = \Theta(n)$ as $3n + 2 \ge 3n$ for all $n \ge 2$, and $3n + 2 \le 4n$ for all $n \ge 2$, so $c_1 = 3$, $c_2 = 4$, and $n_0 = 2$. $3n + 3 = \Theta(n)$; $10n^2 + 4n + 2 = \Theta(n^2)$; $6*2^n + n^2 = \Theta(2^n)$; and $10*\log n + 4 = \Theta(\log n)$. $3n + 2 \ne \Theta(1)$; $3n + 3 \ne \Theta(n^2)$; $10n^2 + 4n + 2 \ne \Theta(n)$; $10n^2 + 4n + 2 \ne \Theta(1)$; $6*2^n + n^2 \ne \Theta(n^2)$; $6*2^n + n^2 \ne \Theta(n^{100})$; and $6*2^n + n^2 \ne \Theta(1)$. □

  - We say that $g(n)$ is an ***asymptotically tight bound*** for $f(n)$

Little-Oh

$o(g(n)) = \{f(n) :$ for any positive constant $c > 0$, there exists a constant $n_0 > 0$ such that $0 \le f(n) < cg(n)$ for all $n \ge n_0\}$.

- The definitions of O-notation and o-notation are similar

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  - $f(n) = O(g(n))$, the bound $0 \le f(n) \le cg(n)$ holds for ***some*** constant $c > 0$
  - $f(n) = o(g(n))$, the bound $0 \le f(n) < cg(n)$ holds for ***all*** constants $c > 0$

- Examples:
  - $2n = o(n^2)$
  - $2n^2 \ne o(n^2)$

Little-Omega

$$\omega(g(n)) = \{ f(n) : \text{ for any positive constant } c > 0, \text{ there exists a constant } n_0 > 0 \text{ such that } 0 \le cg(n) < f(n) \text{ for all } n \ge n_0 \} .$$

- By analogy, ω-notation is to Ω-notation as o-notation is to O-notation

Definition: [Omega] $f(n) = \Omega(g(n))$ (read as "f of n is omega of g of n") iff there exist positive constants c and n_0 such that $f(n) \ge cg(n)$ for all $n, n \ge n_0$. □

- Examples:
  - $\frac{n^2}{2} = \omega(n)$
  - $\frac{n^2}{2} \ne \omega(n^2)$

Questions?

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